Homework 2 for Columbia B9136

Will Ma

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Abstract

Follow the notation from class.

Exercise 1. Prove that for any $T \in \mathbb{N}$,

$$\inf_{\mathbf{F} \in \Delta(\mathbb{R}^T_{\geq 0})} \sup_{\pi \in \Pi} \mathbb{E}_{\mathbf{V} \sim \mathbf{F}} \left[\frac{\pi(\mathbf{V})}{\mathsf{OFF}(\mathbf{V})} \right] = \inf_{\mathbf{F} \in \Delta(\mathbb{R}^T_{\geq 0})} \frac{\sup_{\pi \in \Pi} \mathbb{E}_{\mathbf{V} \sim \mathbf{F}}[\pi(\mathbf{V})]}{\mathbb{E}_{\mathbf{V} \sim \mathbf{F}}[\mathsf{OFF}(\mathbf{V})]}. \tag{0.1}$$

Hint: (0.1) is not true without taking an infimum over $\mathbf{F}$. But given a distribution $\mathbf{F}$ that achieves the infimum on one side, it is possible to re-weight it achieve the infimum on the other side.

Exercise 2. Consider the case where valuations can take any value in $\mathcal{R} := [1, R] \cup \{0\}$, for some $R \geq 1$, but no information about valuations is given otherwise. Consider the generalization where up to k agents can be accepted, for some fixed $k \in \mathbb{N}$. Prove that for any $k \in \mathbb{N}$,

$$\inf_{T \in \mathbb{N}} \sup_{P \in \Delta(\Pi)} \inf_{\mathbf{V} \in \mathcal{R}^T} \frac{P(\mathbf{V})}{\mathsf{OFF}(\mathbf{V})} = \frac{1}{1 + \ln R}.$$

Exercise 3. Consider the Bernoulli optimization problem

$$\begin{aligned} \inf \quad & \Pr\left[\sum_{t=1}^{T} \mathrm{Ber}(x_t) < k\right] \\ \text{s.t.} \quad & \sum_{t=1}^{T} x_t = \lambda \\ & x_t \in [0,1] \qquad\qquad \forall t \in [T] \end{aligned}$$

where $\lambda \geq 0$ is a constant. Fix $k = 2$ and $T = 4$. Characterize the optimal objective value of this Bernoulli optimization problem for all values of $\lambda \in [0, 4]$.
Hint: You may apply the theorem stated in class about Bernoulli Optimization.

Exercise 4. The algorithmic guarantee of $1 - 1/e$ for Prophet Secretary under $k = 1$ crucially relies on a *randomized* static threshold policy, which can set $\Pr[D = 0]$ to exactly $1/e$.

1. Show that *deterministic* static threshold policies (with $\rho = 1$) cannot achieve a guarantee better than $1/2$ relative to $\mathsf{OFF}(\mathbf{F})$, by considering T valuations that are IID with a distribution that is $1/\varepsilon$ w.p. ε/T and 1 otherwise. (Observe that a negative result with IID distributions implies a negative result for Prophet Secretary.)

2. Generalize the previous construction to show that deterministic static threshold policies cannot achieve a guarantee better than $1/2$ under *any* fixed k.

Note: It is easy to verify that in these constructions, $\mathsf{OFF}(\mathbf{F}) = \sup_{\pi \in \Pi} \pi(\mathbf{F})$. Therefore, 1/2 also represents the loss from restricting to deterministic static threshold policies instead of using the optimal online policy.