Homework 4 for Columbia B9136

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Exercise 1. In this problem you are asked to generalize the RANKING algorithm and competitive ratio $1 - 1/e$ for online bipartite matching to the *vertex-weighted* setting, defined as follows. Each item i has a fixed, known reward $r_i > 0$ for getting matched (it can only get matched once). The objective is to maximize the total reward of the final matching.

a) Write the new primal and dual LP's. (This requires a very minor modification.)

b) The generalized RANKING algorithm no longer has a simple random-order interpretation, but is still described by drawing a random seed Y_i uniformly from $[0, 1]$ for each item i. The algorithm is now described by: for each t, match the available neighbor i with highest value of $r_i(1 - p(Y_i))$, where $p(y) = e^{y-1}$ is the same function as before. Small values of Y_i still boost item i's priority. You should now think of item i as generating revenue $r_i p(Y_i)$ when sold, with the utility to the buying agent being $r_i(1 - p(Y_i))$.

   Given this description of the algorithm, define dual variables whose objective is at most the algorithm's value divided by c, where $c = 1 - 1/e$.

c) Argue that the constructed dual solution is feasible in expectation. The paradigm is the same, where we consider edge (i, t) and fix an arbitrary Y_{-i}. Find a generalized way of defining *threshold value* τ so that the following properties hold:

   - if $Y_i < \tau$, then i gets matched sooner or later;
   - β_t will be set to a value at least $r_i(1 - p(\tau))/c$.

   (This is the hardest step, but still follows from the same argument as before, where we have to now be careful that β_t could be set based on an $i' \neq i$ with $r_{i'} \neq r_i$. Also, one new scenario that could arise is $\tau = 0$, i.e. even if $Y_i = 0$ item i is not guaranteed to get matched.)

d) Finish the proof that generalized RANKING is $(1 - 1/e)$-competitive under vertex weights. You can check your steps by setting $r_1 = \cdots = r_n = 1$ and seeing if your proof coincides with the old proof.

Exercise 2. Find a counterexample to "Claim 2" from the analysis of RANKING, in the case of Adwords. In particular, consider the Greedy algorithm, and consider its execution on an instance $\mathcal{I}$ and its parallel execution on an instance $\mathcal{I}'$ that is identical to $\mathcal{I}$ except one advertiser is removed. There is less competition from the advertiser side. Nonetheless, show that for an impression t, it is possible that the revenue earned from it is strictly higher in the parallel universe.