Homework 5 for Columbia B9136

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Exercise 1. Consider the list distribution induced by the Plackett-Luce model. That is, there is an urn will balls $j = 0, 1, \ldots, n$, where each ball j has weight $w_j > 0$, with $w_0 = 1$. Sequentially, a ball is taken out of the urn (sampled without replacement) with probability proportional to its weight (that is, if ball 0 and balls $S \subseteq [n]$ are still in the urn, then the probability of ball j being drawn is $w_j/(w_0 + \sum_{j' \in S} w_{j'})$). The product corresponding to the ball drawn is added to the end of the agent's preference list, and the list ends when the 0 ball is drawn (if balls $3, 5, 7, 0$ are drawn in sequence, then the agent's list ends up being $(3, 5, 7)$).

Consider an agent who draws a random list according to this process, and chooses the most-preferred product from S, or 0 if S contains no products on their list. Prove that the choice probabilities are consistent with the MNL model; i.e. for any assortment S and product $j \in S$, we have

$$\phi(j, S) = \frac{w_j}{1 + \sum_{j' \in S} w_{j'}}.$$

Exercise 2. Consider the random utilities induced by the Thurstone model. That is, the agent draws a random utility[1] in $\mathbb{R}$ for each product $j = 0, \ldots, n$ with distribution $U_j = u_j + \varepsilon_j$, where ε_j is an independent Gumbel(0,1) random variable that is continuously distributed over $\mathbb{R}$ with PDF $f(x) = e^{-(x+e^{-x})}$. Importantly, the utility for the no-purchase option $j = 0$ is also random, with $u_j = 0$. Show that for any assortment S, the choice probabilities follow MNL with weights $w_j = e^{u_j}$:

$$\phi(j, S) = \frac{e^{u_j}}{1 + \sum_{j' \in S} e^{u_{j'}}}.$$

Exercise 3. Consider Nested Logit with dissimilarity parameter $\gamma_i = 0$ for all nests i, where 0^0 is understood to be 0. This induces a random-utility model. Describe this model as a distribution over ranked lists and prove that it is consistent with the choice formulas given by Nested Logit.

Note: The usual Nested Logit model with $\gamma_i > 0$ has no known simple description as a distribution over ranked lists.

Exercise 4.

1. In the *independent demand* model, the agent's list has length at most 1, consisting of product j w.p. λ_j for all $j \in [n]$, where $\sum_j \lambda_j \leq 1$. Show that this list distribution can be captured by the Markov Chain choice model.

2. In the *buy down* model, the agent's list is supported on lists of the form $(n, n-1, \ldots, j)$ for some $j \in [n+1]$. (Recalling that $r_1 \geq \cdots \geq r_n$, list $(n, n-1, \ldots, j)$ can be interpreted as the agent having willingness-to-pay r_j and always preferring cheaper products over more expensive ones.) Show that this list distribution can be captured by the Markov Chain choice model.

[1] This should be interpreted as the utility under the fixed price of the product.

3. Consider a list that is drawn according to some independent demand model half the time, and drawn according to some buy down model the other half the time. Construct an example showing that this *cannot* be captured by a Markov Chain choice model.