Homework 9 for Columbia B9136

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Abstract

Warning: this Homework is somewhat harder than previous ones; please start early!

Exercise 1. Consider Newsvendor where the demand distribution F is continuous over [0,1], with a PDF that is at least γ over $[0,1]$, where γ is a constant in $(0,1]$. Prove that the Regret $\mathbb{E}[L(\hat{a})-L(a^*)]$ vanishes at the faster rate of $O(1/N)$, by extending three different proofs from class. (You must do all three parts; proving it once is not enough.)

1. In our first direct proof (Lecture 8), we showed that $L(\hat{a})-L(a^*) \le \sup_{a\in[0,1]}|F(a)-\hat{F}(a)|$ for general distributions over [0,1]. Given the added assumption of lower-bounded PDF, show that
$$L(\hat{a})-L(a^*) \le \frac{1}{\gamma}\sup_{a\in[0,1]}(F(a)-\hat{F}(a))^2.$$
From here apply DKW and the same integration trick to show that $\mathbb{E}[L(\hat{a})-L(a^*)] = O(1/N)$.

2. In our On-Average Stability proof (Lecture 8), we showed for general distributions over $[0,1]$ that
$$\mathbb{E}\left[\xi^{(\ell+1)}-\xi^{(\ell)}\right] = \int_0^1 \binom{N}{\ell} F(\xi)^\ell(1-F(\xi))^{N-\ell}d\xi \tag{0.1}$$
and proceeded to establish $O(1/\sqrt{N})$ regret by taking a supremum over $\xi\in[0,1]$.

   Now that we have the lower-bounded PDF assumption, we can do a finer analysis. Note that the integrand $F(\xi)^\ell(1-F(\xi))^{N-\ell}$ is maximized when $F(\xi)=\ell/N$. Let $\xi^*=F^{-1}(\ell/N)$. Note that (0.1) can be re-written as
$$\begin{aligned}&\binom{N}{\ell}\left(\int_0^{\xi^*} F(\xi)^\ell(1-F(\xi))^{N-\ell}d\xi+\int_{\xi^*}^1 F(\xi)^\ell(1-F(\xi))^{N-\ell}d\xi\right)\\ &\le \binom{N}{\ell}\Bigg(\int_{\xi^*-\frac{\ell}{\gamma N}}^{\xi^*}(\frac{\ell}{N}-\gamma(\xi^*-\xi))^\ell(\frac{N-\ell}{N}+\gamma(\xi^*-\xi))^{N-\ell}d\xi\\ &+\int_{\xi^*}^{\xi^*+\frac{N-\ell}{\gamma N}}(\frac{\ell}{N}+\gamma(\xi-\xi^*))^\ell(\frac{N-\ell}{N}-\gamma(\xi-\xi^*))^{N-\ell}d\xi\Bigg)\end{aligned}$$
Finish the analysis of this term to show that the regret is $O(1/N)$.

3. Specializing our second direct proof (Lecture 9) to bounded demands over [0,1], we get
$$\mathbb{E}[L(\hat{a})-L(a^*)] \le \int_0^1 |q-F(\xi)|\exp(-2N(q-F(\xi))^2)d\xi.$$

Split this integral into three parts:

$$\int_0^1 = \int_0^{a^* - \frac{1}{2\gamma\sqrt{N}}} + \int_{a^* - \frac{1}{2\gamma\sqrt{N}}}^{a^* + \frac{1}{2\gamma\sqrt{N}}} + \int_{a^* + \frac{1}{2\gamma\sqrt{N}}}^1$$

(making sure to cover boundary cases where $a^* - \frac{1}{2\gamma\sqrt{N}} < 0$, etc.) and prove that all three parts are $O(1/N)$. A hint for the two outer parts: show that the function $g(x) = xe^{-2Nx^2}$ is decreasing over $x \geq \frac{1}{2\sqrt{N}}$, after which you can argue that $|q - F(\xi)| \geq \gamma|a^* - \xi|$ using the lower bound on the PDF.

Exercise 2. In class we derived algorithms A with regret $\mathbb{E}[L(A) - L(a^*)] = O(\frac{1}{(1-q)\sqrt{N}})$, for all $F \in \Delta(\mathbb{R}_{\geq 0})$ with mean promised to be at most 1. The denominator approaches 0 as $q \to 1$, which means that if q can change with N, then the bound no longer vanishes as $N \to \infty$. Prove that this is unavoidable, in that there exists a constant $c > 0$, such that for any N, there exists $q \in [0, 1]$ such that any data-driven algorithm A (knowing q) must have $\mathbb{E}[L(A) - L(a^*)] \geq c$, on some distribution $F \in \Delta(\mathbb{R}_{\geq 0})$ with mean at most 1.

Hint: You will have to use the same framework as the $\Omega(1/\sqrt{N})$ lower bound that we proved in class. In the example I constructed to solve this exercise, upper-bounding $\mathrm{TV}(P^N, Q^N)$ for the two possible distributions P, Q is much easier and does not need to go through Hellinger distance.